INTERACTIVE TOPSIS ALGORITHM FOR A SPECIAL TYPE OF LINEAR FRACTIONAL VECTOR OPTIMIZATION PROBLEMS

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Abstract

In this paper, TOPSIS (Technique for Order Preference by Similarity Ideal Solution) is extended for solving Linear Fractional Vector Optimization problems (LFVOP) of a special type, where such problems have block angular structure of the constraints. In order to obtain a compromise (satisfactory) solution to the above problems using the TOPSIS approach, a modified formulas for the distance function from the positive ideal solution (PIS) and the distance function from the negative ideal solution (NIS) are proposed. An interactive decision making algorithm for generating a compromise (satisfactory) solution through TOPSIS approach is provided where the decision maker (DM) is asked to specify the degree α and the relative importance of the objectives. Finally, a numerical example is given to clarify the main results developed in the paper.

Keywords: TOPSIS, Interactive decision making, Vector optimization problems, Linear fractional programming, Block angular structure, Angular and dual-angular structure, Fuzzy set theory, Compromise (satisfactory) solution, Positive ideal solution, Negative ideal solution.

1.0 INTRODUCTION

Fractional programming has attracted the attention of many researchers in the past. The main reason for interest in fractional programming stems from the fact that programming models could better fit the real problems if we consider optimization of ratio between the physical and/or economic quantities. Literature survey reveals wide applications of fractional programming in different areas ranging from engineering to economics.

There are many familiar structures for large scale optimization problems [2,3,4,5,22, 37,38, 38,47,48], such as: (i) the block angular structure, and (ii) angular and dual-angular structure to the constraints, and several kinds of decomposition methods for linear and nonlinear programming problems with those structures have been proposed in [20, 36]. An algorithm for parametric large scale integer linear multiple objective decision-making problems with block angular structure has been presented in [2].

Recently a significant number of studies have indeed been reported on single and multiple objective fractional linear and nonlinear programming problems [3, 5, 12, 14, 18, 19,20, 25,26,27,40,42,43,44,45,46,49].

Ammar [12] discussed Dinkelbach’s global optimization approach for finding the global maximum of the fractional programming problem. Based on this idea, the author gave several characterizations of the solution set of a convex–concave fractional programs.

Caballero and Hernández [14] introduced a new method to estimate the weakly efficient set for the multiobjective linear fractional programming problem. A useful and new approach to solve
fractional programming with absolute-value functions (FP-A) has been proposed by Chang [18]. Later on, Chang in [19] introduced an approximate approach to reaching as close as possible an optimal solution of the fractional programming problem with absolute value function (FP-A).

Dutta et al. [25] developed a method for optimizing multiobjective linear fractional programming problem which yields always an efficient solution.

Husain and Jabeen [28] derived necessary and sufficient optimality conditions for a continuous - time fractional minmax programming problem. In another work, Husain et al. [29] introduced necessary and sufficient optimality conditions for a nondifferentiable fractional minimax programming problem.

Moreover, sufficient optimality conditions for a nonlinear multiple objective fractional programming problem involving η-semidifferentiable type I-preinvex and related functions have been derived by Mishra et al. [40].

A goal programming (GP) procedure for fuzzy multiobjective linear fractional programming problems has been suggested by Pal et al. [42]. Preda [43] presented necessary and sufficient optimality conditions for a nonlinear fractional multiple objective programming problem involving η-semidifferentiable functions.

Ravi and Reddy [44] have modeled chemical process plant operations planning in an oil refinery as fuzzy linear fractional multiple goal programming problem.

Saad and Sharif [45] proposed a solution procedure to solve the chance-constrained integer linear fractional programming problem.

Stancu-Minasian and Pop [49] pointed out certain shortcomings in the work of Dutta et al. [25] and give the correct proof of theorem which validates the obtaining of the efficient solutions.

M. A. Abo–Sinna and T. H. M. Abou-El-Enien [7] introduce an algorithm for solving large scale multiple objective decision making (LSMODM) problems by use of TOPSIS.

M. A. Abo-Sinna and A. H. Amer [8] focus on multiobjective large scale nonlinear programming (MOLSNLP) problems with block angular structure. They extend TOPSIS approach to solve (MOLSNLP) problems.

G. Tzeng et al. [50] compare and apply TOPSIS and VIKOR to determine the best compromise alternative fuel mode. The result shows that the hybrid electric bus is the most suitable substitute bus for Taiwan urban areas in the short and median term. But, if the cruising distance of electric bus extends to an acceptable range, the pure electric bus could be the best alternative.

M. A. Abo–Sinna and T. H. M. Abou-El-Enien [9] extend TOPSIS for solving interactive large scale multiple Objective programming problems involving fuzzy parameters. These fuzzy parameters are characterized as fuzzy numbers. For such problems, the α-Pareto optimality is introduced by extending the ordinary Pareto optimality on the basis of the α-Level sets of fuzzy numbers. An interactive fuzzy decision making algorithm for generating α-Pareto optimal solution through TOPSIS approach is provided where the decision maker (DM) is asked to specify the degree α and the relative importance of objectives. Finally, a numerical example is given to clarify the main results developed in this paper.

O. M. Saad and T. H. M. Abou-El-Enien [46] are concerned with the solution of integer linear fractional multiple objective programming problems (ILFMOP) of a special type, where such problems have block angular structure of the constraints. The presented solution algorithm is based mainly upon a combination of Charens - Cooper transformation coupled with Branch and Bound method together with Rosen's Partitioning procedure. To demonstrate the proposed algorithm, a numerical example to clarify the main results and the developed theory in the paper is provided.

T. H. M. Abou-El-Enien and O. M. Saad [3] are concerned with solving large scale linear fractional multiple objective programming (LSLFMO) problems with chance constraints. Chance constraints involve random parameters in the right-hand sides. These random right-hand sides are considered to be statistically independent random variable. The main features of the proposed solution procedure are based on chance-constrained technique, Charens and Cooper transformation and the Rosen's partitioning procedure. An illustrative
numerical example is given to clarify the main results developed in the paper.

T. H. M. Abou-El-Enien [4] focuses on the solution of a Large Scale Integer Linear Vector Optimization Problems with chance constraints (CHLSILVOP) of a special type through the Technique for Order Preference by Similarity Ideal Solution (TOPSIS) approach, where such problems has block angular structure of the constraints. Chance constraints involve random parameters in the right hand sides. TOPSIS is extended to solve CHLSILVOP with constraints of block angular structure. Compromise (TOPSIS) control minimizes the measure of distance, provided that the closest solution should have the shortest distance from the positive ideal solution (PIS) as well as the longest distance from the negative ideal solution (NIS). As the measure of “closeness” $d_p$-metric is used. Thus, He reduces a $k$-dimensional objective space to a two-dimensional space by a first-order compromise procedure. The concept of a membership function of fuzzy set theory is used to represent the satisfaction level for both criteria. Moreover, He derives a single objective large scale Integer programming problem using the max–min operator for the second-order compromise operation. Also, An interactive decision making algorithm for generating integer Pareto optimal (compromise) solution for CHLSILVOP through TOPSIS approach is provided where the decision maker (DM) is asked to specify the relative importance of objectives. Finally, a numerical example is given to clarify the main results developed in this paper.

After the publication of TOPSIS approach [30,34], the subsequent works in this area of optimization have been numerous (see f. i. [1,4,5,6,7,8,9,10,11,15,16,17,24,31,32,33,34,35,50]. In the present paper, a solution algorithm is suggested for the solution of linear fractional vector optimization problems (LFVOP) with block angular structure of the constraints through TOPSIS approach.

The paper is organized as follows: In the following section, the problem formulation of LFVOP which has block angular structure is formulated. An algorithm is described in finite steps for solving the problem of concern is proposed in Section 3. For the sake of illustration, a numerical example is provided in Section 4. Finally, the paper is concluded in Section 5.

2.0 PROBLEM FORMULATION

Consider the following linear fractional vector optimization problem with a block angular structure of the constraints as [3,46]:

\[
\text{Minimize} \quad (F_1(X), F_2(X), \ldots, F_k(X)) \quad (1-a)
\]

subject to

\[
X \in M = \left\{ X \in \mathbb{R}^m \left| \sum_{j=1}^q A_j X_j \leq v_s \right. \right\} \quad (1-b)
\]

where the $i^{th}$ objective function can be written as follows:

\[
F_i(X) = \frac{\sum_{j=1}^q f_{ij}(X)}{\sum_{j=1}^q f_{ij}(X)} = \frac{\sum_{j=1}^q c_{ij} X_j + \gamma_i}{\sum_{j=1}^q b_{ij} X_j + \beta_i}, \quad i=1,2,\ldots,k \quad (2)
\]

and

$\gamma_i$ and $\beta_i$ are constants, $i=1,2,\ldots,k$;

$k$ : the number of objective functions,

$q$ : the number of subproblems,

$m$ : the number of constraints,

$n$ : the number of variables,

$n_j$ : the number of variables of the $j^{th}$ subproblem, $j=1,2,\ldots,q > 1$

$m_o$ : the number of the common constraints represented by: $\sum_{j=1}^q A_j X_j \leq v_0$,

$m_j$ : the number of independent constraints of the $j^{th}$ subproblem represented by:

$B_j X_j \leq \nu_j, \quad j=1,2,\ldots,q, \quad q > 1$
\[ R : \text{the set of all real numbers,} \]
\[ X : \text{an n-dimensional column vector of variables,} \]
\[ X_j : \text{an } n_j \text{-dimensional column vector of variables for the } j^{th} \text{ subproblem, } j=1,2,\ldots,q \>
\[ A_j : \text{an } (m_o \times n_j) \text{ coefficient matrix,} \]
\[ B_j : \text{an } (m_j \times n_j) \text{ coefficient matrix,} \]
\[ v_o : \text{an } m_o \text{-dimensional column vector of right-hand sides of the common constraints whose elements are constants,} \]
\[ v_j : \text{an } m_j \text{-dimensional column vector of independent constraints right-hand sides whose elements are the constants of the constraints for the } j^{th} \text{ subproblem, } j=1,2,\ldots,q \>
\[ C_{ij} : \text{an } n_j \text{-dimensional row vector for the } j^{th} \text{ subproblem in the } i^{th} \text{ objective function,} \]
\[ D_{ij} : \text{an } n_j \text{-dimensional row vector for the } j^{th} \text{ subproblem in the } i^{th} \text{ objective function,} \]
\[ K = \{1,2,\ldots,k\}, \quad N = \{1,2,\ldots,n\} \quad \text{and} \quad R^o = \{x=(x_1, x_2,\ldots,x_n) \in R, \: i \in N\}. \]

Furthermore, we assume that \( \sum_{j=1}^{q} D_j X_j + \beta \) is everywhere positive on the set of non-negative numbers.

Consequently, using Charnes-Cooper transformation method \([20]\) by making the variable change:
\[ \mu = \frac{1}{\sum_{j=1}^{q} D_j X_j + \beta_j} \] 

(3-a)

with the additional variable changes
\[ Y_j = X_j \mu, \quad j = 1,2,\ldots,q, q > 1 \]

(3-b)

then under these changes, problem (1) is equivalent to the following problem:

\[ \text{Minimize} \quad (F_1(Y, \mu), F_2(Y, \mu),\ldots., F_k(Y, \mu)) \] 

subject to
\[ X \in \mathbb{R}^n \quad | \sum_{j=1}^{q} A_j Y_j - v_o \mu \leq 0, \] 
\[ \sum_{j=1}^{q} D_j Y_j + \beta \mu = 1, \] 
\[ B_j Y_j - v_j \mu \leq 0, \] 
\[ \mu > 0, \] 
\[ Y_j \geq 0, \] 
\[ j = 1,2,\ldots,q, q > 1 \]

(4-b)

where equation (2) can be rewritten as follows :

\[ F_i(X) = F_i(Y, \mu) = \sum_{j=1}^{q} F_{ij}(Y, \mu) \]

\[ = \sum_{j=1}^{q} C_{ij} Y_j + Y_j \mu, \quad i = 1,2,\ldots,k, \]

Now, problem (4-a)-(4-b) has angular and dual-angular structure \([21,37]\) and can be solved using the Rosen's partitioning procedure \([21,37]\) to find its optimal solution \( Y^*_j \), which will give the optimal solution \( X^*_j = Y^*_j / \mu^* \) to problem (1).

### 3.0 SOME BASIC CONCEPTS OF DISTANCE MEASURES

The compromise programming approach \([5,26,30,31,34,35,52]\) has been developed to perform multiple objective decision making problems, reducing the set of nondominated solutions \([50]\). The compromise solutions are those which are the closest by some distance measure to the ideal one.

The point \( F_i(X^*) = \sum_{j=1}^{q} F_{ij}(X^*) \) in the criteria space is called the ideal point (reference point). As the measure of “closeness”, \( d_p \)-metric is used. The \( d_p \)-metric defines the distance between two points \( F_i(X) = \sum_{j=1}^{q} F_{ij}(X) \) and \( F_i(X^*) = \sum_{j=1}^{q} F_{ij}(X^*) \) (the reference point) in k-dimensional space \([40]\) as:

\[ d_p = \left( \sum_{i=1}^{k} (F_i^* - F_i)^p \right)^{\frac{1}{p}} \]
\[
= \left( \sum_{i=1}^{q} \left( \sum_{j=1}^{q} F_{ij}^* - \sum_{j=1}^{q} F_{ij} \right)^{p} \right)^{\frac{1}{p}}
\]

where \( p \geq 1 \).

Unfortunately, because of the incommensurability among objectives, it is impossible to directly use the above distance family. To remove the effects of the incommensurability, we need to normalize the distance family of equation (5) by using the reference point \([5,26,30,31,34,35,52]\) as:

\[
d_p = \left( \sum_{i=1}^{q} \left( \frac{\sum_{j=1}^{q} F_{ij}^* - \sum_{j=1}^{q} F_{ij}}{\sum_{j=1}^{q} F_{ij}^*} \right)^{p} \right)^{\frac{1}{p}}
\]

where \( p \geq 1 \).

To obtain a compromise solution for problem (1), the global criteria method \([30, 53]\) for large scale problems uses the distance family of equation (5) by the ideal solution being the reference point. The problem becomes how to solve the following auxiliary problem:

**Minimize** \( \text{d}_p \) :

\[
\left( \sum_{i=1}^{q} \left( \frac{\sum_{j=1}^{q} F_{ij}^* (X^*) - \sum_{j=1}^{q} F_{ij} (X)}{\sum_{j=1}^{q} F_{ij} (X^*)} \right)^{p} \right)^{\frac{1}{p}}
\]

where \( X^* \) is the PIS and \( p = 1, 2, \ldots, \infty \).

Usually, the solutions based on PIS are different from the solutions based on NIS. Thus, both \( PIS(F^*) \) and \( NIS(F^-) \) can be used to normalize the distance family and obtain \([5]\):

\[
d_p = \left( \sum_{i=1}^{q} \left( \frac{\sum_{j=1}^{q} F_{ij}^* - \sum_{j=1}^{q} F_{ij}}{\sum_{j=1}^{q} F_{ij}^* - \sum_{j=1}^{q} F_{ij}} \right)^{p} \right)^{\frac{1}{p}}
\]

where \( p \geq 1 \).

In this study, the concept of TOPSIS is extended to obtain a Pareto Optimal (compromise) solution for LFVOP problems.

**4.0 TOPSIS FOR LFVOP:**

Consider the following LFVOP \([5]\):

\[
\text{Maximize/Minimize } (F_1(X), F_2(X), \ldots, F_k(X))
\]

subject to

\[
X \in M^f
\]

where

\[
\sum_{j=1}^{q} F_{ij} (X) : \text{Objective Function for Maximization,}
\]

\[
\sum_{j=1}^{q} F_{ij} (X) : \text{Objective Function for Minimization,}
\]

In order to use the distance family of equation (8) to resolve problem (9), we must first find \( PIS(F^*) \) and \( NIS(F^-) \) which are \([5]\):

\[ F^* = \left( \sum_{j=1}^{q} F_{ij} (X^*) \right) \text{ or } \left( \sum_{j=1}^{q} F_{ij} (X) \right) \]

\[ \forall t \text{ (and v)} \]

\[ F^- = \left( \sum_{j=1}^{q} F_{ij} (X) \right) \text{ or } \left( \sum_{j=1}^{q} F_{ij} (X) \right) \]

\[ \forall t \text{ (and v)} \]

where \( K = K_1 \cup K_2 \).

\[ F^* = (F_{1^*}, F_{2^*}, \ldots, F_{k^*}) \text{ and } F^- = (F_{1^-}, F_{2^-}, \ldots, F_{k^-}) \] are the individual positive (negative) ideal solutions.
Using the PIS and the NIS, we obtain the following distance functions from them, respectively:

\[
\begin{align*}
\text{d}^{\text{PIS}}_p &= \left( \sum_{t \in \mathcal{T}_1} w_t^p \left( \frac{\sum_{j=1}^{q} f_{ij} - \sum_{j=1}^{q} f_{ij}(X)}{\sum_{j=1}^{q} f_{ij}} \right)^p \right) + \\
&\quad \left( \sum_{t \in \mathcal{T}_2} w_t^p \left( \frac{\sum_{j=1}^{q} f_{ij}(X) - \sum_{j=1}^{q} f_{ij}}{\sum_{j=1}^{q} f_{ij}} \right)^p \right)^{1/p} \\
&= \frac{1}{\left( \sum_{t \in \mathcal{T}_1} w_t^p \left( \frac{\sum_{j=1}^{q} f_{ij} - \sum_{j=1}^{q} f_{ij}(X)}{\sum_{j=1}^{q} f_{ij}} \right)^p \right) + \\
&\quad \left( \sum_{t \in \mathcal{T}_2} w_t^p \left( \frac{\sum_{j=1}^{q} f_{ij}(X) - \sum_{j=1}^{q} f_{ij}}{\sum_{j=1}^{q} f_{ij}} \right)^p \right)^{1/p}} \\
\text{d}^{\text{NIS}}_p &= \left( \sum_{t \in \mathcal{T}_1} w_t^p \left( \frac{\sum_{j=1}^{q} f_{ij} - \sum_{j=1}^{q} f_{ij}(X)}{\sum_{j=1}^{q} f_{ij}} \right)^p \right) + \\
&\quad \left( \sum_{t \in \mathcal{T}_2} w_t^p \left( \frac{\sum_{j=1}^{q} f_{ij}(X) - \sum_{j=1}^{q} f_{ij}}{\sum_{j=1}^{q} f_{ij}} \right)^p \right)^{1/p} \\
&= \frac{1}{\left( \sum_{t \in \mathcal{T}_1} w_t^p \left( \frac{\sum_{j=1}^{q} f_{ij} - \sum_{j=1}^{q} f_{ij}(X)}{\sum_{j=1}^{q} f_{ij}} \right)^p \right) + \\
&\quad \left( \sum_{t \in \mathcal{T}_2} w_t^p \left( \frac{\sum_{j=1}^{q} f_{ij}(X) - \sum_{j=1}^{q} f_{ij}}{\sum_{j=1}^{q} f_{ij}} \right)^p \right)^{1/p}} \\
\end{align*}
\]

(11-a) and

(11-b)

where \( w_t = 1, 2, \ldots, k \), are the relative importance (weights) of objectives, and \( p = 1, 2, \ldots, \infty \).

In order to obtain a compromise solution, transfer problem (9) into the following bi-objective problem with two commensurable (but often conflicting) objectives [5]:

**Minimize** \( d^{\text{PIS}}_p (X) \)

**Maximize** \( d^{\text{NIS}}_p (X) \)

subject to

\[ X \in M^l \]

(12)

where \( p = 1, 2, \ldots, \infty \).

Since these two objectives are usually conflicting to each other, we can simultaneously obtain their individual optima. Thus, we can use membership functions to represent these individual optima. Assume that the membership functions (\( \mu_1(X) \) and \( \mu_2(X) \)) of two objective functions are linear. Then, based on the preference concept, we assign a larger degree to the one with shorter distance from the PIS for \( \mu_1(X) \) and assign a larger degree to the one with farther distance from NIS for \( \mu_2(X) \). Therefore, as shown in figure (1), \( \mu_1(X) \) and \( \mu_2(X) \) can be obtained as the following [5,54]:

\[
\begin{align*}
\mu_1(X) &= \begin{cases} 
1, & \text{if } d^{\text{PIS}}_p (X) < \left( d^{\text{PIS}}_p \right)^*, \\
1 - \frac{d^{\text{PIS}}_p (X) - \left( d^{\text{PIS}}_p \right)^*}{\left( d^{\text{PIS}}_p \right)^* - \left( d^{\text{PIS}}_p \right)^*}, & \text{if } d^{\text{PIS}}_p (X) \geq d^{\text{PIS}}_p \geq \left( d^{\text{PIS}}_p \right)^*, \\
0, & \text{if } d^{\text{PIS}}_p (X) > \left( d^{\text{PIS}}_p \right)^*, 
\end{cases} \\
\end{align*}
\]

(13-a)

\[
\begin{align*}
\mu_2(X) &= \begin{cases} 
1, & \text{if } d^{\text{NIS}}_p (X) > \left( d^{\text{NIS}}_p \right)^*, \\
1 - \frac{d^{\text{NIS}}_p (X) - \left( d^{\text{NIS}}_p \right)^*}{\left( d^{\text{NIS}}_p \right)^* - \left( d^{\text{NIS}}_p \right)^*}, & \text{if } d^{\text{NIS}}_p (X) \leq d^{\text{NIS}}_p \leq \left( d^{\text{NIS}}_p \right)^*, \\
0, & \text{if } d^{\text{NIS}}_p (X) < \left( d^{\text{NIS}}_p \right)^*, 
\end{cases} \\
\end{align*}
\]

(13-b)

where

\[
\begin{align*}
\left( d^{\text{PIS}}_p \right)^* &= \min_{X \in M^l} d^{\text{PIS}}_p (X) \text{ and the solution is } X^{\text{PIS}}, \\
\left( d^{\text{NIS}}_p \right)^* &= \max_{X \in M^l} d^{\text{NIS}}_p (X) \text{ and the solution is } X^{\text{NIS}}, \\
\left( d^{\text{PIS}}_p \right)^* &= d^{\text{PIS}}_p (X^{\text{NIS}}) \text{ and } \left( d^{\text{NIS}}_p \right)^* = d^{\text{NIS}}_p (X^{\text{PIS}}). \\
\end{align*}
\]

Now, by applying the max-min decision model which is proposed by Bellman and Zadeh [13] and extended by Zimmermann [54], we can resolve problem (12). The satisfying decision, \( X^* \), may be obtained by solving the following model:

\[
\begin{align*}
\mu_D(X^*) &= \max_{X \in M^l} \{ \min \left( \mu_1(X), \mu_2(X) \right) \} \\
\end{align*}
\]

(14)
Finally, if \( \delta = \min(\mu_1(X), \mu_2(X)) \), the model (14) is equivalent to the form of Tchebycheff model [23], which is equivalent to the following model:

\[
\begin{align*}
\text{Maximize } & \delta, & (15 - a) \\
\text{subject to } & \mu_1(X) \geq \delta, & (15 - b) \\
& \mu_2(X) \geq \delta, & (15 - c) \\
& X \in M', & \delta \in [0,1] & (15 - d)
\end{align*}
\]

where \( \delta \) is the satisfactory level for both criteria of the shortest distance from the PIS and the farthest distance from the NIS.

5.0 THE ALGORITHM OF TOPSIS FOR SOLVING LFVOP:

Thus, we can introduce the following algorithm to generate a set of Pareto Optimal (compromise) solutions for the LFVOP:

**The algorithm (Alg-I):**

**Step 1.** Formulate LFVOP (1) which has linear fractional objective functions as in Eq. (2) and block angular structure.

**Step 2.** Use the Charnes-Cooper transformation [20] by making the variable change \( \mu = \frac{1}{\sum_{j=1}^{q} \delta_j x_j + \beta_j} \) with the additional variable change \( y_j = x_j \mu \) \( j = 1, 2, \ldots, q, q > 1 \) to rewrite problem (1) in the form of problem (4).

**Step 3.** Transform problem (4) to the form of problem (9).

**Step 4.** Construct the PIS payoff table of problem (9) by using Rosen's partitioning procedure [21, 37]. Thus, \( F^* = (F^*_1, F^*_2, \ldots, F^*_k) \), the individual positive ideal solutions are obtained.

**Step 5.** Construct the NIS payoff table of problem (9) by using Rosen's partitioning procedure [21, 37]. Thus, \( F^- = \)
Step 6. Use equations (10 & 11) and steps (4 & 5) to construct $d_p^{PIS}$ and $d_p^{NIS}$.

Step 7. Transform problem (9) to problem (12).

Step 8. Ask the DM to select $p^*$ where

$$\sum_{i=1}^{k} w_i = 1$$

Step 9. Ask the DM to select $\mu = \frac{1}{4x_1 + 3x_2 + 3}$, therefore the above problem will take the following form:

Maximize $F_1(x, \mu)$

Minimize $F_2(x, \mu)$

subject to

$$x_1 + x_2 \leq 4,$$
$$x_1 \leq 3,$$
$$x_1 \geq 1,$$
$$x_2 \leq 1.5,$$
$$x_2 \geq 0.5,$$
$$x_1, x_2 \geq 0,$$

where

$$F_1(x) = \frac{3x_1 + 5x_2}{4x_1 + 3x_2 + 3},$$
$$F_2(x) = \frac{7x_1 + 2x_2}{4x_1 + 3x_2 + 3}$$

Step 10. Construct the payoff table of problem (12):

At $p=1$, use Rosen's partitioning procedure [21, 37].

At $p \geq 2$, use the generalized reduced gradient method [37, 38] and obtain:

$$d_p^- = (d_p^{PIS}^- - d_p^{NIS}^-),$$
$$d_p^+ = (d_p^{PIS}^+ - d_p^{NIS}^+).$$

Step 11. Construct problem (15) by the use of the membership functions (13).


Step 13. If the solution of problem (15) yields optimal solution, then go to step 14. Otherwise go to step 15.

Step 14. If the DM is satisfied with the current solution, go to step 15. Otherwise, go to step 8.

Step 15. Stop.

6.0 AN ILLUSTRATIVE NUMERICAL EXAMPLE

In what follows, we provide a numerical example to illustrate the solution algorithm described in the previous section. For this purpose, let us consider the following linear fractional vector optimization problem which has a block angular structure of the constraints as:

Minimize $F_1(x)$

Maximize $F_2(x)$,

subject to

$$x_1 + x_2 \leq 4,$$
$$x_1 \leq 3,$$
$$x_1 \geq 1,$$
$$x_2 \leq 1.5,$$
$$x_2 \geq 0.5,$$
$$x_1, x_2 \geq 0,$$

where

$$F_1(x) = \frac{3x_1 + 5x_2}{4x_1 + 3x_2 + 3},$$
$$F_2(x) = \frac{7x_1 + 2x_2}{4x_1 + 3x_2 + 3}$$

Assume that $\mu = \frac{1}{4x_1 + 3x_2 + 3}$.

Now, making the variable changes:

$$y_1 = x_1 \mu \quad \text{and} \quad y_2 = x_2 \mu$$

Thus, the above problem can be rewritten as follows:
Maximize \( F_1(y, \mu) \)
Minimize \( F_2(y, \mu) \),
subject to \( y_1 + y_2 - 4\mu \leq 0, \)
\( 4y_1 + 3y_2 + 3\mu = 1, \)
\( y_1 - 3\mu \leq 0, \quad y_1 - \mu \geq 0, \)
\( y_2 - 1.5\mu \leq 0, \quad y_2 - 0.5\mu \geq 0, \)
\( y_1, y_2 \geq 0 \text{ and } \mu > 0, \)

This problem has angular and dual-angular structure.

Table (1): PIS payoff table of problem (18)

<table>
<thead>
<tr>
<th></th>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximize</td>
<td>0.91304*</td>
<td>0.86957</td>
<td>0.08696</td>
<td>0.13043</td>
<td>0.08696</td>
</tr>
<tr>
<td>Minimize</td>
<td>0.91304</td>
<td>0.86957*</td>
<td>0.08696</td>
<td>0.13043</td>
<td>0.08696</td>
</tr>
</tbody>
</table>

PIS: \( f^* = (0.91304, 0.86957) \)

Table (2): NIS payoff table of problem (16)

<table>
<thead>
<tr>
<th></th>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimize</td>
<td>0.64706</td>
<td>0.94119</td>
<td>0.11765</td>
<td>0.05882</td>
<td>0.11765</td>
</tr>
<tr>
<td>Maximize</td>
<td>0.6971</td>
<td>1.33333</td>
<td>0.18182</td>
<td>0.0303</td>
<td>0.06061</td>
</tr>
</tbody>
</table>

NIS: \( f^- = (0.64706, 1.33333) \)

Next, we compute equation (11) and obtain the following equations:

\[
d^P_{IS} = \left[ w_1^P \left( \frac{f_1 - 0.64706}{0.91304 - 0.64706} \right)^p + w_2^P \left( \frac{f_2 - 0.86957}{1.33333 - 0.86957} \right)^p \right]^{1/p}
\]

Thus, problem (12) is obtained.

In order to get numerical solutions, let us assume that \( w_1^P = w_2^P = 0.5 \) and (1) at \( p=1, \)

Table (3): PIS payoff table of problem (12), when \( p=2, \)

<table>
<thead>
<tr>
<th></th>
<th>( d^P_{IS} )</th>
<th>( d^N_{IS} )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( \mu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimize</td>
<td>0.000004</td>
<td>0.7071</td>
<td>0.087</td>
<td>0.1304</td>
<td>0.087</td>
</tr>
<tr>
<td>Maximize</td>
<td>0.000004</td>
<td>0.7071</td>
<td>0.087</td>
<td>0.1304</td>
<td>0.087</td>
</tr>
</tbody>
</table>

\( \delta = (0.000004, 0.7071) \), \( d^N_{IS} = 0.000004 \geq 0 \)

Now, it is easy to use problem (15) to formulate the following problem:

Maximize \( \delta \)
Subject to:

\( X \in M', \delta \in [0,1], d^P_{IS} - 0.000004 \geq 0, \)
\( 0.7071 - d^N_{IS} \geq 0 \)

The maximum "satisfactory level" (\( \delta = 1 \)) is achieved for the solution when
\[ y_1 = 0.087, \ y_2 = 0.1304, \ \mu = 0.087, \ x_1 = 1 \text{ and } \ x_2 = 1.49885. \]

7.0 CONCLUSIONS

In this paper, a TOPSIS approach has been extended to solve LFVOP. The LFVOP using TOPSIS approach provides an effective way to find the compromise (satisfactory) solution of such problems. Generally TOPSIS, provides a broader principle of compromise for solving Vector Optimization Problems. It transfers k-objectives (criteria), which are conflicting and non-commensurable, into two objectives (the shortest distance from the PIS and the longest distance from the NIS), which are commensurable and most of time conflicting. Then, the bi-objective problem can be solved by using membership functions of fuzzy set theory to represent the satisfaction level for both criteria and obtain TOPSIS, compromise solution by a second-order compromise. The max-min operator is then considered as a suitable one to resolve the conflict between the new criteria (the shortest distance from the PIS and the longest distance from the NIS).

Also, in this paper, an algorithm of generating Pareto Optimal (compromise) solutions of LFVOP problem has been presented. It is based on the decomposition algorithm of LFVOP with block angular structure via TOPSIS approach for \( p = 1 \) and Generalized reduced gradient method for \( p \geq 2 \). This algorithm combines LFVOP and TOPSIS approach to obtain TOPSIS's compromise solution of the problem. Finally, a numerical illustrative example clarified the various aspects of both the solution concept and the proposed algorithm.

REFERENCES


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